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A Leray Theorem for the Generalization to Operads of Hopf Algebras with Divided Powers

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INTRODUCTION

Let A be a connected commutative graded Hopf algebra over a field of characteristic zero, let $I(A)$ be its augmentation ideal, and let $Q(A)$ be the set of indecomposables of A. The classical Leray theorem asserts that any section $Q(A) \overset{s}{\to} I(A)$ induces an isomorphism from the free commutative algebra over $Q(A)$ to A [MM, Theorem 7.5]. The section s can be chosen to be canonical by means of the weight-decomposition techniques of [P1] and [P2].

To generalize the theorem to the prime characteristic case, it is necessary to make a further hypothesis on the Hopf algebra structure. Namely, one has to deal with Hopf algebras with divided powers. André proved that a connected Hopf algebra with divided powers A is isomorphic to the universal enveloping coalgebra of a graded Lie coalgebra [A, p. 20]. The corresponding "Leray theorem," that is, the result that A is free as an algebra with divided powers, was stated and proved by Sjödin [S]. It is also a consequence of the work of André. In fact, Block [Bl1, p. 309] noted that using the results of André, the very short proof of the Leray theorem in [MM] does go through in the divided power case. In the same paper, Block extended the Leray theorem to irreducible divided power Hopf algebras (graded or not, of any characteristic).

We concentrate in this article on connected graded algebras. Connected commutative graded Hopf algebras are cogroups in the category of connected commutative graded algebras, so that the Leray theorem states that cogroups in this category of algebras are free algebras. We are interested

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in generalizing the theorem to various kinds of algebras over fields of finite characteristic. Let us look at an example. In the category of groups, Kan [K] proved that cogroups are free groups. Let now G be a group and p be a prime. The group G has a decreasing filtration: the mod-p lower central series of G (see, e.g., [Cu, p. 167]),

$$\cdots \subset \Gamma_{n+1}G \subset \cdots \subset \Gamma_1 G = G.$$

Each $\Gamma_r G$ is the subgroup of G generated by the elements $[\gamma_1, \ldots, \gamma_s]^{p^t}$, where $[\ ,\ldots,\ ]$ is the iterated commutator $[\cdots[\ ,\ ], \ldots,\ ]$, $\gamma_i \in G$, and $s \cdot p^t \geq r$. Zassenhaus [Z] proved that if G is free, the quotients of the mod-p lower central series of G form a free p-restricted Lie algebra, $L(G)$, generated by $G/\Gamma_2 G$. A corollary of the Zassenhaus result is that the p-restricted Lie algebra $L(G * G')$ associated to the free product of two free groups G and G' is the coproduct (in the category of p-restricted Lie algebras) of $L(G)$ and $L(G')$. In particular, if G is a cogroup in the category of groups, $L(G)$ is both a cogroup in the category of p-restricted Lie algebras and a free p-restricted Lie algebra. We will show that the corresponding "Leray theorem" is true in general: cogroups in the category of (connected graded) restricted Lie algebras are free restricted Lie algebras.

Our starting point is the theory of operads. Most of the usual algebras are algebras over an operad: associative algebras, commutative algebras, Lie algebras, Poisson algebras, and so forth. There are also many graded algebras over operads which come from the theory of iterated loop spaces, such as the n-Lie algebras or the n-braid algebras [KM, I.5, I.6]. In the finite characteristic case, the structures involved are, for example, higher analogues of restricted Lie algebras [Co, pp. 220–224]. One of the first interesting results in the study of iterated loop spaces is the computation of $H_*(\Omega\Sigma X, K)$ (K being a field, Σ being the suspension functor, and Ω being the based loop space functor). The Bott–Samelson theorem states that $H_*(\Omega\Sigma X, K)$ is a free associative algebra (there are analogous results when $\Omega\Sigma$ is replaced by $\Omega^n\Sigma^n$; see [Co]). Berstein observed that this result can be deduced from a suitable Leray theorem: cogroups in the category of associative algebras (over any field) are free algebras [B]. Our results should clarify why it is natural to assume the existence of operations such as divided powers or restricted power maps (for restricted Lie algebras and their generalizations) to extend Berstein's result to general operads over a field of finite characteristic.

If the characteristic of the ground field is zero, a Leray theorem for algebras over operads is known: a cogroup in a category of (graded connected) algebras over an operad is a free algebra over this operad. It was known for associative algebras [B] and dual Leibniz algebras [O1], and was proven in full generality by Fresse using Cartier duality [F1]. Another proof of this theorem, reminiscent of the proof of the Leray theorem in [MM], has been given by Oudom in his thesis [O2].

In this paper, we prove that, over a field of any characteristic, cogroups in categories of algebras with divided operations over an operad (with divided symmetries, in the terminology of [F2]) are free as algebras with divided operations. This includes the case of algebras with divided powers and restricted Lie algebras.

Over a field of characteristic zero, algebras with divided operations identify with (usual) algebras over an operad, so that our proof also provides in that case a new demonstration of the Leray theorem for algebras over operads.

We use freely the language of triples and (co)groups in categories [McL]. A pair of maps (f, g) from X to Y is reflexive if there is a map Δ from Y to X such that $f \circ \Delta = \mathrm{id}_Y = g \circ \Delta$. We recall from [L] that, in (suitable) categories of algebras over a triple, coproducts are created by coequalizers of reflexive pairs.

1. DEFINITIONS AND WEIGHT DECOMPOSITIONS

We first recall from [M1] and [M2] the definition of algebras over an operad. See also the seminal article [GK] for more details on the role of operads in algebra and for some standard definitions, in particular, the definition of the operads As, Com, and Lie, which are, respectively, the operads of associative, commutative, and Lie algebras.

An operad is a sequence of symmetric group representations which encode the n-ary operations on a given family of algebras. Precisely, let K be a field. An operad is given by a family of vector spaces $\mathscr{P} = (\mathscr{P}(n))_{n \geq 0}$, a unit map η: $K \to \mathscr{P}(1)$, a right action of the symmetric group S_j on $\mathscr{P}(j)$ for each j, and structure maps

$$\gamma: \mathscr{P}(k) \otimes \mathscr{P}(j_1) \otimes \cdots \otimes \mathscr{P}(j_k) \to \mathscr{P}(j)$$

for $k \geq 1$ and $j \geq 0$, where $\sum_{i=1}^{k} j_i = j$. The structure maps are required to be associative, unital, and equivariant in a natural sense. For example, $\mathrm{As}(n) = K[S_n]$, $\mathrm{Com}(n) = K$, and $\mathrm{Lie}(n)$ is the Lie representation [GK].

To each operad $\mathscr{P}$ is canonically associated a triple (or monad). An algebra over $\mathscr{P}$ or $\mathscr{P}$-algebra is an algebra over this triple (see [M2, p. 19, Monadic reinterpretation of algebras]). In other terms, a $\mathscr{P}$-algebra is a vector space A together with maps μ_n: $\mathscr{P}(n) \otimes_{S_n} A^{\otimes n} \to A$, which have to be associative, unital, and equivariant in a natural sense.

In this article, we restrict our attention to operads for which $\mathscr{P}(0) = 0$ and $\mathscr{P}(1) = K$. Actually, we work in the category $\mathscr{P}^{\mathrm{red}}$ of reduced $\mathbb{N}$-graded $\mathscr{P}$-algebras, that is, $\mathbb{N}$-graded $\mathscr{P}$-algebras $A = \bigoplus_{k=0}^{\infty} A_k$ such that $A_0 = 0$. This category has coproducts, created by coequalizers of reflexive pairs [L]. The coproduct of A and A' in $\mathscr{P}^{\mathrm{red}}$ will be written $A \coprod A'$. An object of $\mathscr{P}^{\mathrm{red}}$ has a canonical decreasing filtration $(\mathscr{F}^n A)_{n\in\mathbb{N}}$ defined by

$$\mathscr{F}^n A := \mathrm{Im}\left(\oplus \mu_k \colon \bigoplus_{k\geq n} \mathscr{P}(k) \otimes_{S_k} A^{\otimes k} \to A \right).$$

The associated bi-graded $\mathscr{P}$-algebra is written

$$\mathrm{Gr}\, A = \bigoplus_{k\in\mathbb{N}^*} \mathrm{Gr}_k A = \bigoplus_{k\in\mathbb{N}^*} \mathscr{F}^k A / \mathscr{F}^{k+1} A.$$

The first term of $\mathrm{Gr}\, A$, $\mathrm{Gr}_1 A = A/\mathscr{F}^2 A$, by definition, is the vector space $Q(A)$ of *indecomposables* of A. If f is a $\mathscr{P}$-algebra endomorphism of A, we write $Q(f)$ for the induced endomorphism of $Q(A)$.

For short, we call a cogroup in the category of reduced $\mathbb{N}$-graded $\mathscr{P}$-algebras a *co-H-$\mathscr{P}$-algebra* (the H is for Hopf). For example, if $\mathscr{P}$ is the operad As of associative algebras, a co-H-As-algebra A is a cogroup in the category of associative algebras. In particular, the coproduct is a map from A to the free product of two copies of A and not a map from A to $A \otimes A$, so that a co-H-As-algebra is not a Hopf algebra in the usual sense.

If A is a co-H-$\mathscr{P}$-algebra, we write Δ (resp. Δ^k) for the coproduct map

$$A \xrightarrow{\Delta} A \coprod A$$

(resp. the iterated coproduct $A \xrightarrow{\Delta^k} A^{\coprod k}$).

LEMMA 1.1. *Let $A \in Ob(\mathscr{P}^{\mathrm{red}})$ and let f be a $\mathscr{P}$-algebra endomorphism of A such that $Q(f) = k$, k in the prime field of K, $k \notin \{0,1\}$. If* char $K = p$ *there is a canonical weight decomposition, $A = \bigoplus_{i=1}^{p-1} A^{(i)}$, where the component of degree n of $A^{(i)}$, $A_n^{(i)}$, is the eigenspace associated to the eigenvalue k^i of $f^{p^{n-1}}$.*

Assume by induction that $\bigoplus_{m\leq n} A_m$ decomposes into eigenspaces for the eigenvalues $k, \dots, k^{p-1}$ of $f^{p^{n-1}}$. Then, since f is a $\mathscr{P}$-algebra endomorphism of A, $(f^{p^{n-1}})^{p-1}$ acts identically on the degree $n+1$ component of $\mathscr{F}^2 A$.

Let $x \in A_{n+1}$. Since $Q(f) = k$, we have

$$\exists z \in \mathscr{F}^2 A, \qquad \left(f^{p^{n-1}}\right)^{p-1}(x) = x + z.$$

Moreover,

$$\left(\left(f^{p^{n-1}}\right)^{p-1}\right)^{p}(x) = x + pz = x$$

and $X^{p-1} - 1$ is an annulation polynomial for f^{p^n}. Hence the lemma.

PROPOSITION 1.2. *If* char $K = p$, *a* co-H-$\mathscr{P}$-*algebra* A *has a canonical decomposition into weight components* $A = \oplus_{i=1}^{p-1} A^{(i)}$.

The situation is precisely the same as for commutative connected Hopf algebras [P1]. Namely, the composite Ψ^k of the iterated coproduct map $\Delta^k\colon A \to A^{\amalg k}$ and of the canonical map $A^{\amalg k} \to A$ satisfies the conditions of Lemma 1.1. The proposition still holds when A is a comonoid (comultiplicative object, coassociative, with a co-unit) in the category of reduced $\mathbb{N}$-graded $\mathscr{P}$-algebras.

Proposition 1.2 implies that in the vector space generated by the Ψ^k there exist canonical elements $(e^i)_{i \in [1, p-1]}$ which are idempotents for the composition of endomorphisms of A. These idempotents are the projections $A \to A^{(i)}$. They can be computed as linear combinations of the Ψ^k using Lagrange interpolation formulas. They generalize idempotents which were first introduced by Gerstenhaber and Schack, who studied the mod-p shuffle bi-algebra and gave explicit formulas for the corresponding idempotents (which can be expressed in this case as linear combinations of permutations) [GS, p. 271].

2. A LERAY THEOREM FOR ALGEBRAS WITH DIVIDED OPERATIONS

The Leray theorem (a commutative connected Hopf algebra is a free commutative algebra) does not hold in finite characteristic. However, it holds if the Hopf algebra has divided powers (that is, set endomorphisms which behave like $x^k/k!$; see below for a precise definition of algebras with divided powers): any connected graded Hopf algebra with divided powers is free as an algebra with divided powers [S].

A free algebra with divided powers over a vector space V identifies with the subalgebra of symmetric tensors of the shuffle algebra over V [Bl2]. The divided powers are given by $\gamma^k(x) := x \otimes \cdots \otimes x = x^{\otimes k}$ if $x \in V$. In particular, any algebra with divided powers A is canonically provided with a map

$$(A^{\otimes n})^{S_n} \to A.$$

In the language of triples, the free algebra with divided powers functor

$$\mathscr{T}(V) := \bigoplus_{n \in \mathbb{N}^*} (V^{\otimes n})^{S_n}$$

is associated with a triple, and algebras with divided powers can be defined as the algebras over this triple.

This definition was generalized to algebras over operads in [F2]. For any operad $\mathscr{P}$, the functor "free $\mathscr{P}$-algebra with divided operations" is defined by

$$\hat{\mathscr{P}}(V) := \bigoplus_n \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)^{S_n}.$$

There is an associated triple written $\langle \hat{\mathscr{P}}, \eta, \nu \rangle$, where η is given by the isomorphism $V \cong \mathscr{P}(1) \otimes V$ (recall that we have assumed that our operads satisfy $\mathscr{P}(0) = 0$ and $\mathscr{P}(1) = K$). We give explicit formulas for the natural transformation ν from $\hat{\mathscr{P}}\hat{\mathscr{P}}$ to $\hat{\mathscr{P}}$ by generalizing the construction of the free algebra with divided powers. Compare with the definition of ν in [F2].

Let M be an S_n-module and M^{S_n} the corresponding module of invariants. The symmetrization map is the map Σ_n, from M to M^{S_n} defined by $\Sigma_n(x) := \sum_{\sigma \in S_n} \sigma \cdot x$. For any vector space V and any operad $\mathscr{P}$, the symmetrization map induces a map

$$\Sigma_n \colon \mathscr{P}(n) \otimes V^{\otimes n} \to \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)^{S_n}$$

(resp. Σ from $\bigoplus_n \mathscr{P}(n) \otimes V^{\otimes n}$ to $\bigoplus_n (\mathscr{P}(n) \otimes V^{\otimes n})^{S_n}$).

Over a field of characteristic zero, the map $\nu(V)$ from $\hat{\mathscr{P}}\hat{\mathscr{P}}(V)$ to $\hat{\mathscr{P}}(V)$ is defined as follows. The structure map

$$\gamma \colon \mathscr{P}(n) \otimes \left(\mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_n)\right) \to \mathscr{P}(m_1 + \cdots + m_n),$$

induces a map

$$\begin{aligned}\gamma_n \colon \mathscr{P}(n) \otimes \left(\left(\mathscr{P}(m_1) \otimes V^{\otimes m_1}\right) \otimes \cdots \otimes \left(\mathscr{P}(m_n) \otimes V^{\otimes m_n}\right)\right) \\ \to \mathscr{P}(m_1 + \cdots + m_n) \otimes V^{\otimes m_1 + \cdots + m_n}\end{aligned}$$

(defined by first grouping together the $V^{\otimes m_i}$ on the right and letting γ act on $\mathscr{P}(n) \otimes (\mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_n))$). We write I for the restriction of this map γ_n to the subspace $(\mathscr{P}(n) \otimes ((\mathscr{P}(m_1) \otimes V^{\otimes m_1})^{S_{m_1}} \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n})^{S_{m_n}}))^{S_n}$. The restriction of $\nu(V)$ to $(\mathscr{P}(n) \otimes ((\mathscr{P}(m_1) \otimes V^{\otimes m_1})^{S_{m_1}} \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n})^{S_{m_n}}))^{S_n}$ is the "divided operation" defined by

$$\nu(V) := \frac{1}{n!m_1! \cdots m_n!} \cdot \Sigma_{m_1 + \cdots + m_n} \circ I.$$

Over an arbitrary field, $n!m_1! \cdots m_n!$ is in general not invertible. The construction follows in this case from the equivariance condition in the definition of operads. Let $X := \{(m_1, \ldots, m_n),\ m_j \in \mathbb{N}^*\}$ and let

$$x = \sum_{i \in X} p_i \otimes x_i^1 \otimes \cdots \otimes x_i^n,$$

where $x \in \bigoplus_{(m_1, \ldots, m_n) \in X} (\mathscr{P}(n) \otimes ((\mathscr{P}(m_1) \otimes V^{\otimes m_1})^{S_{m_1}} \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n})^{S_{m_n}}))^{S_n}$.

The action of S_{m_i} on $\mathscr{P}(m_i) \otimes V^{\otimes m_i}$ induces an action of $S_{m_1} \times \cdots \times S_{m_n}$ on $\mathscr{P}(n) \otimes ((\mathscr{P}(m_1) \otimes V^{\otimes m_1}) \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n}))$ and the structure map γ is equivariant for this action of $S_{m_1} \times \cdots \times S_{m_n}$. That is, the following diagram commutes, where $\sigma_i \in S_{n_i}$ and $\sigma_1 \oplus \cdots \oplus \sigma_n$ is the block sum [M1, p. 2]:

$$\begin{array}{ccc}
\mathscr{P}(n) \otimes (\mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_n)) & \xrightarrow{\mathrm{id} \otimes \sigma_1 \otimes \cdots \otimes \sigma_n} & \mathscr{P}(n) \otimes (\mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_n)) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathscr{P}(m_1 + \cdots + m_n) & \xrightarrow{\sigma_1 \oplus \cdots \oplus \sigma_n} & \mathscr{P}(m_1 + \cdots + m_n)
\end{array}$$

In particular, the image by γ_n of $\mathscr{P}(n) \otimes ((\mathscr{P}(m_1) \otimes V^{\otimes m_1})^{S_{m_1}} \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n})^{S_{m_n}})$ is invariant under the action of the subgroup $S_{m_1} \times \cdots \times S_{m_n}$ of $S_{m_1 + \cdots + m_n}$. We define $S(m_1, \ldots, m_n)$ and Ξ by

$$S(m_1, \ldots, m_n) := S_{m_1 + \cdots + m_n} / S_{m_1} \times \cdots \times S_{m_n},$$

and

$$\Xi(x) := \Big\{ \big(\sigma, p_i \otimes x_i^1 \otimes \cdots \otimes x_i^n\big) \Big| i = (m_1, \ldots, m_n) \in X, \\ \sigma \in S(m_1, \ldots, m_n) \Big\}.$$

Next, S_n acts on the right on $\mathscr{P}(n)$ and by block permutation on $\bigoplus_{m_1, \ldots, m_n} (\mathscr{P}(m_1) \otimes V^{\otimes m_1}) \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n})$. For the corresponding action of S_n on $\bigoplus_{m_1, \ldots, m_n} \mathscr{P}(n) \otimes ((\mathscr{P}(m_1) \otimes V^{\otimes m_1}) \otimes \cdots \otimes (\mathscr{P}(m_n) \otimes V^{\otimes m_n}))$, the map γ_n is equivariant. Precisely, the following diagram commutes, where $\sigma \in S_n$ and $\sigma(m_1, \ldots, m_n) \in S_{m_1 + \cdots + m_n}$ permutes n blocks as σ permutes n elements (see [M1]):

$$\begin{array}{ccc}
\mathscr{P}(n) \otimes (\mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_n)) & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathscr{P}(n) \otimes (\mathscr{P}(m_{\sigma(1)}) \otimes \cdots \otimes \mathscr{P}(m_{\sigma(n)})) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathscr{P}(m_1 + \cdots + m_n) & \xrightarrow{\sigma(m_{\sigma(1)}, \ldots, m_{\sigma(n)})} & \mathscr{P}(m_1 + \cdots + m_n)
\end{array}$$

This action by block permutation "commutes" with the action of $S_{m_1} \times \cdots \times S_{m_n}$. That is, if $\sigma \in S_n$ and $(\beta_1, \dots, \beta_n) \in S_{m_1} \times \cdots \times S_{m_n}$, then

$$(\beta_{\sigma^{-1}(1)}, \dots, \beta_{\sigma^{-1}(n)}) \circ (\sigma^{-1} \otimes \sigma) = (\sigma^{-1} \otimes \sigma) \circ (\beta_1, \dots, \beta_n).$$

Therefore the action of S_n by block permutation induces a (free) right action of S_n on $\coprod_X S(m_1, \dots, m_n)$, each $\sigma \in S_n$ defining a bijection between $S(m_1, \dots, m_n)$ and $S(m_{\sigma(1)}, \dots, m_{\sigma(n)})$.

At last, let us define an action of S_n on $\Xi(x)$ by

$$\forall \beta \in S_n,$$
$$\beta\left(\sigma, p_i \otimes x_i^1 \otimes \cdots \otimes x_i^n\right) := \left(\sigma \cdot \beta^{-1}, \beta\left(p_i \otimes x_i^1 \otimes \cdots \otimes x_i^n\right)\right).$$

This action is well defined (since x is S_n-invariant) and is free. Moreover, since γ is S_n equivariant, any two elements in the same S_n-orbit have the same image under the map

$$\Pi : \Xi(x) \to \mathscr{P}(m_1 + \cdots + m_n) \otimes V^{\otimes m_1 + \cdots + m_n},$$
$$\Pi\left(\sigma, p_i \otimes x_i^1 \otimes \cdots \otimes x_i^n\right) := \sigma \circ \gamma_n\left(p_i \otimes x_i^1 \otimes \cdots \otimes x_i^n\right).$$

Therefore, Π factorizes by the quotient set $\Xi(x)/S_n$, and the map ν,

$$\nu(x) := \sum_{z \in \Xi(x)/S_n} \Pi(z),$$

from

$$\bigoplus_{(m_1, \dots, m_n) \in X} \left(\mathscr{P}(n) \otimes \left(\left(\mathscr{P}(m_1) \otimes V^{\otimes m_1}\right)^{S_{m_1}} \otimes \cdots \otimes \left(\mathscr{P}(m_n) \otimes V^{\otimes m_n}\right)^{S_{m_n}}\right)\right)^{S_n}$$

to $(\mathscr{P}(m_1 + \cdots + m_n) \otimes V^{\otimes m_1 + \cdots + m_n})^{S_{m_1 + \cdots + m_n}}$, is well defined.

Compare with the definition of divided powers in [A, pp. 24–26]. Of course, if char $K = 0$, this definition agrees with the definition which was given above. That $(\hat{\mathscr{P}}, \eta, \nu)$ is indeed a triple follows from the general properties (associativity, equivariance, unit) of the operad $\mathscr{P}$.

Definition 2.1. The $\mathscr{P}$ algebras with divided operations or $\hat{\mathscr{P}}$-algebras are the algebras over the triple $\langle \hat{\mathscr{P}}, \eta, \nu \rangle$.

Let us consider some classical algebras and operads and the corresponding algebras with divided operations (see also [F2]). If $\mathscr{P} = \mathrm{Com}$ (commutative algebras), the $\widehat{\mathrm{Com}}$-algebras are the algebras with divided powers (see above). If $\mathscr{P} = \mathrm{As}$ (associative algebras), since $\mathrm{As}(n) = K[S_n]$, we have $\mathrm{As}(n) \otimes_{S_n} V^{\otimes n} \cong (\mathrm{As}(n) \otimes V^{\otimes n})^{S_n}$, so that As and $\widehat{\mathrm{As}}$-algebras identify. If char K is not 2 and $\mathscr{P} = \mathrm{Lie}$ (Lie algebras), the category of $\widehat{\mathrm{Lie}}$-algebras is also a well-known category. Recall the definition of restricted Lie algebras (see [MM, Chap. 6] and [J, Chap. 5, Sect. 7]). A (p-) restricted Lie algebra is a Lie algebra of characteristic p in which there is a "p-power map" $x \mapsto x^{[p]}$ such that

$$\text{(i)} \qquad (\alpha x)^{[p]} = \alpha^p x^{[p]},$$

$$\text{(ii)} \qquad (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y),$$

$$\text{(iii)} \qquad [xy^{[p]}] = x(\mathrm{ad}\, y)^p,$$

where α is a scalar and $i \cdot s_i(x, y)$ is the coefficient of λ^{i-1} in $x(\mathrm{ad}(\lambda x + y))^{p-1}$ (we write ad for the adjoint action). These relations correspond to well-known identities in associative algebras of characteristic p such as

$$[\cdots[[x,y],y]\cdots,y] = [x, y^p],$$

with p brackets on the left, or

$$[\cdots[[x,y],y]\cdots,y] = \sum_{i=0}^{p-1} y^{p-1-i} x y^i,$$

with $p - 1$ brackets on the left (see [J, Chap. 5, Sect. 7]). The last identity shows that, *formally*, we have

$$x^{[p]} = \frac{1}{p}[\cdots[[x,x],x]\cdots,x],$$

in the same way that in the divided power case the divided power of an element x is *formally* equal to $x^p/p!$. These properties are closely related to the definition of the natural transformation ν from $\widehat{\mathrm{LieLie}}$ to $\widehat{\mathrm{Lie}}$: any $\widehat{\mathrm{Lie}}$-algebra is a restricted Lie algebra. In fact, it is proven in [F2] that $\widehat{\mathrm{Lie}}$-algebras and restricted Lie algebras identify. Precisely, the free $\widehat{\mathrm{Lie}}$-algebra over a vector space V is canonically isomorphic to the restricted Lie algebra of primitive elements in the free associative algebra over V, which is known to be isomorphic to the free restricted Lie algebra over V.

Let us return to the general case. As usual, coproducts of $\hat{\mathscr{P}}$-algebras exist and are created by coequalizers of reflexive pairs [L]: the coproduct of $(A_i)_{i \in I}$ is the coequalizer of the pair

$$\begin{tikzcd}
\left(\hat{\mathscr{P}}\left(\oplus \hat{\mathscr{P}}(A_i)\right), \mu\right) \arrow[r, "\hat{\mathscr{P}}(\cdots \hat{\mathscr{P}}(j_i) \cdots)"] \arrow[d, "="] & \left(\hat{\mathscr{P}}\hat{\mathscr{P}}(\oplus A_i), \mu\right) \arrow[r, "\mu"] & \left(\hat{\mathscr{P}}(\oplus A_i), \mu\right) \\
\left(\hat{\mathscr{P}}\left(\oplus \hat{\mathscr{P}}(A_i)\right), \mu\right) \arrow[rr, "\hat{\mathscr{P}}(\oplus \alpha_i)"] & & \left(\hat{\mathscr{P}}(\oplus A_i), \mu\right) \arrow[u, "="']
\end{tikzcd}$$

where j_i is the inclusion of A_i into $\oplus A_i$ and α_i is the canonical map from $\hat{\mathscr{P}}(A_i)$ to A_i.

In particular, the coproduct $A \coprod A = A^{\coprod 2}$ (resp. $A \coprod A \cdots \coprod A = A^{\coprod n}$) is a quotient of the free $\hat{\mathscr{P}}$-algebra over $A \oplus A$ (resp. $A \oplus \cdots \oplus A = A^{\oplus n}$).

Observe that the map $\hat{\mathscr{P}}(\oplus \eta_{A_i})$,

$$\hat{\mathscr{P}}(\oplus A_i) \to \hat{\mathscr{P}}\left(\oplus \hat{\mathscr{P}}(A_i)\right),$$

makes the pair reflexive.

In case char $K = 0$, the transfer map from coinvariants to invariants (induced by the symmetrization map) induces isomorphisms,

$$\mathscr{P}(n) \otimes_{S_n} V^{\otimes n} \to \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)^{S_n},$$

and also an isomorphism between the free $\mathscr{P}$-algebra and the free $\hat{\mathscr{P}}$-algebra. In particular, the categories of $\mathscr{P}$ and $\hat{\mathscr{P}}$-algebras are equivalent. The Leray theorem (cogroups are free algebras) is known in this case [F1]. It is also know for $\mathscr{P}$-algebras (in any characteristic) when the $\mathscr{P}(n)$ are free as S_n-modules, since the proof of Berstein for associative algebras can be adapted in this case [O1, Remark 3.7, p. 126]. But, of course, this is a very particular case and most of the usual operads (such as Com or Lie) are not given by free S_n-modules. In general, a Leray theorem can be proven only for cogroups in categories of $\hat{\mathscr{P}}$-algebras (see Theorem 2.4 below).

A $\hat{\mathscr{P}}$-algebra has a natural filtration:

$$\mathscr{F}^k A := \operatorname{Im}\left(\bigoplus_{n \geq k} \left(\mathscr{P}(n) \otimes A^{\otimes n}\right)^{S_n} \to A \right).$$

DEFINITION 2.2. The first component $Q(A) := \mathrm{Gr}_1 A$ of the bi-graded $\hat{\mathscr{P}}$-algebra

$$(\mathrm{Gr}_k A)_{k \in N^*}, \mathrm{Gr}_k A := \mathscr{F}^k A / \mathscr{F}^{k+1} A$$

is the set of indecomposables of A.

DEFINITION 2.3. A co-H-$\hat{\mathscr{P}}$-algebra is a cogroup in the category of $\mathbb{N}$-graded reduced $\hat{\mathscr{P}}$-algebras.

THEOREM 2.4. *Let A be a* co-H-$\hat{\mathscr{P}}$*-algebra. Any section s of the projection map $A \to Q(A)$ induces a $\hat{\mathscr{P}}$-algebra isomorphism*

$$\hat{s}\colon \hat{\mathscr{P}}(Q(A)) \overset{\cong}{\to} A.$$

In fact, the theorem still holds when A is a comonoid in the category of reduced $\mathbb{N}$-graded $\mathscr{P}$-algebras (the proof does not require the existence of an inverse for the comultiplication).

Our proof combines the ideas developed in [P2] with those of Sjödin [S]. As for the demonstration of the Cartier–Milnor–Moore theorem in [P2], we study the combinatorics of iterated coproducts.

Let us choose an ordered basis $(x_i)_{i \in I}$ of homogeneous elements of $Q(A)$, which we identify with their images by s in A. The ordering is chosen so that $x_i < x_j$ if $\deg x_i < \deg x_j$.

DEFINITION 2.5. A basis $(y_j)_{j \in J}$ of the free $\hat{\mathscr{P}}$-algebra over $Q(A)$ is said to be monomial if its elements are "monomials" in the following sense: $\forall j \in J$, $\exists (i_1, \dots, i_k) \in I$, $x_{i_1} \geq \cdots \geq x_{i_k}$, $\exists (\sigma_1, \dots, \sigma_l) \in S_k$, $\exists (p_1, \dots, p_l) \in \mathscr{P}(k)|$,

$$y_j = \sum_{m=1}^{l} p_m \otimes x_{i_{\sigma_m(1)}} \otimes \cdots \otimes x_{i_{\sigma_m(k)}}.$$

The unique ordered decreasing sequence $(x_{i_1}, \dots, x_{i_k})$ associated to an element y_j of a monomial basis of $\hat{\mathscr{P}}(Q(A))$ is written $S(y_j)$.

Monomial bases of $\hat{\mathscr{P}}(Q(A))$ clearly exist. Let us choose an ordered monomial basis $(y_j)_{j \in J}$, ordered so that $y_j < y_k$ if $S(y_j) < S(y_k)$ for the lexicographic order of words over the ordered set $(x_i)_{i \in I}$.

LEMMA 2.6. *For any sequence $A_1, \dots, A_n$ of $\hat{\mathscr{P}}$-algebras, there is a canonical map*

$$A_1 \amalg \cdots \amalg A_n \overset{p_n}{\to} \mathscr{P}(n) \otimes Q(A_1) \otimes \cdots \otimes Q(A_n).$$

Observe first that there are canonical projections

$$\left(\hat{\mathscr{P}}(\oplus A_i), \nu\right) \to \left(\hat{\mathscr{P}}(\oplus Q(A_i)), \nu\right)$$

and

$$\hat{\mathscr{P}}(\oplus Q(A_i)) = \bigoplus_n \left(\mathscr{P}(n) \otimes \left(\bigoplus Q(A_i)\right)^{\otimes n}\right)^{S_n}$$
$$\to \left(\mathscr{P}(n) \otimes \left(\bigoplus Q(A_i)\right)^{\otimes n}\right)^{S_n}$$
$$\to \mathscr{P}(n) \otimes Q(A_1) \otimes \cdots \otimes Q(A_n).$$

Write p_n their composite. The lemma follows from the identity

$$p_n \circ \left(\nu \circ \hat{\mathscr{P}}\left(\cdots \hat{\mathscr{P}}(j_i) \cdots\right) - \hat{\mathscr{P}}(\oplus \alpha_i)\right) = 0,$$

which is clear since the image of $\nu \circ \hat{\mathscr{P}}(\cdots \hat{\mathscr{P}}(j_i) \cdots) - \hat{\mathscr{P}}(\oplus \alpha_i)$ involves only tensor products $p_k \otimes x_1 \otimes \cdots \otimes x_k$, where either some x_i belong to some $\mathrm{Gr}_2 A_j$ or two of the x_i's belong to the same A_j (this is because the map $\hat{\mathscr{P}}(\oplus \eta_{A_i})$ makes the pair $(\nu \circ \hat{\mathscr{P}}(\cdots \hat{\mathscr{P}}(j_i) \cdots), \hat{\mathscr{P}}(\oplus \alpha_i))$ reflexive).

Let now A be a co-H-$\hat{\mathscr{P}}$-algebra and write Δ^n for the iterated coproduct map (provided by the cogroup structure on A),

$$A \to A^{\amalg n} = A_1 \amalg \cdots \amalg A_n,$$

where each A_i is a copy of A.

COROLLARY 2.7. *Let A be a co-H-$\hat{\mathscr{P}}$-algebra. There is a canonical map*

$$A \xrightarrow{\pi_n} \mathscr{P}(n) \otimes Q(A)^{\otimes n}$$

given by $\pi_n := p_n \circ \Delta^n$.

Recall that we have chosen for each $k \in N^*$ a basis of monomials $(y_j)_{j \in J_k}$ of $(\mathscr{P}(k) \otimes Q(A)^{\otimes k})^{S_k}$, and let us complete this basis to a basis $(y_j)_{j \in J_k} \cup (z_i)_{i \in K_k}$ of $\mathscr{P}(k) \otimes Q(A)^{\otimes k}$, where the z_i's are chosen to be monomials; that is, each z_i involves only combinations with coefficients in $\mathscr{P}(k)$ of permutations of a given tensor product $x_{i_1} \otimes \cdots \otimes x_{i_k}$.

Write ϕ_j for the projection onto the y_j axis orthogonal to the vector space generated by $(y_{j'})_{j' \in J_k - \{j\}} \cup (z_i)_{i \in K_k}$.

LEMMA 2.8. *Let $y_j \in (\mathscr{P}(n) \otimes Q(A)^{\otimes n})^{S_n}$ belong to the chosen basis. The evaluation of $\phi_j \circ \pi_n \circ \hat{s}$ on $y_{j'}$ is equal to zero if $j' < j$.*

Let us first consider the case $S(y_{j'}) < S(y_j)$. Since we have chosen the ordering of the x_i's compatible with the grading, the composite $\pi_n \circ \hat{s}$ only involves tensor products $p_n \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ where, up to any reordering, the word $x_{i_1} \cdots x_{i_n}$ is strictly less than $S(y_j)$ for the lexicographical order, so that $\phi_j \circ \pi_n \circ \hat{s}(y_{j'}) = 0$.

Let us then consider the case $S(y_{j'}) = S(y_j)$. The vector $y_{j'}$ decomposes as

$$y_{j'} = \sum_{k=1}^{l} p_k \otimes x_{i_{\sigma_k(1)}} \otimes \cdots \otimes x_{i_{\sigma_k(n)}},$$

where $\sigma_k \in S_n$ and $(x_{i_1}, \dots, x_{i_n}) = S(y_{j'})$. Since the iterated coproduct is a $\hat{\mathscr{P}}$-algebra map, we have

$$\Delta^n \circ \hat{s}(y_{j'}) = \Delta^n \circ \nu_n \left(\sum_{k=1}^{l} p_k \otimes x_{i_{\sigma_k(1)}} \otimes \cdots \otimes x_{i_{\sigma_k(n)}} \right)$$

$$\left(\text{where } \nu_n \text{ is the structure map } \left(\mathscr{P}(n) \otimes A^{\otimes n}\right)^{S_n} \to A\right)$$

$$= \nu_n \left(\sum_{k=1}^{l} p_k \otimes \Delta^n(x_{i_{\sigma_k(1)}}) \otimes \cdots \otimes \Delta^n(x_{i_{\sigma_k(n)}}) \right)$$

$$\left(\text{where } \nu_n \text{ is the structure map } \left(\mathscr{P}(n) \otimes (A^{\amalg n})^{\otimes n}\right)^{S_n} \to A^{\amalg n}\right).$$

But in $A^{\amalg n} = A_1 \amalg \cdots \amalg A_n$, $A_i \cong A$, we have

$$\Delta^n(x_i) = x_i^{(1)} + \cdots + x_i^{(n)} + z_i,$$

where $x_i^{(l)}$ is a notation for the copy of x_i in A_l and z_i is a linear combination of classes of tensor products involving only elements of degree strictly less than the degree of x_i. Finally, since the tensor products involving the z_i's must cancel under the action of $\phi_j \circ p_n$, the composite $\phi_j \circ \pi_n \circ \hat{s}(y_{j'})$ is equal to

$$\phi_j \circ p_n \circ \nu_n \left(\sum_{k=1}^{l} p_k \otimes \left(\sum_{i=1}^{n} x_{i_{\sigma_k(1)}}^{(i)} \right) \otimes \cdots \otimes \left(\sum_{i=1}^{n} x_{i_{\sigma_k(n)}}^{(i)} \right) \right)$$

$$= \phi_j \left(\sum_{k=1}^{l} p_k \otimes x_{i_{\sigma_k(1)}}^{(1)} \otimes \cdots \otimes x_{i_{\sigma_k(n)}}^{(n)} \right) = \phi_j(y_{j'}) = 0.$$

The same computation yields the following lemma.

LEMMA 2.9. *We have the identity*

$$\phi_j \circ \pi_n \circ \hat{s}(y_j) = y_j.$$

We are now in position to prove the theorem. Since s is a section of the projection on the indecomposables, the map $\hat{s}\colon \hat{\mathscr{P}}(Q(A)) \to A$ is surjective. Suppose there exists a (finite nontrivial) linear combination $\sum_{j \in J} \alpha_j y_j$

such that $\hat{s}(\sum_{j \in J} \alpha_j y_j) = 0$. Let i be the highest $j \in J$ such that $\alpha_j \neq 0$. Then, by the previous lemmas,

$$\phi_i \circ \pi_n \circ \hat{s}\left(\sum_{j \in J} \alpha_j y_j \right) = \alpha_i y_i \neq 0;$$

hence we have a contradiction, and the proof of Theorem 2.4 also follows.

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